Spectral analysis of the massless Dirac operator on a 3-manifold

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The talk will give an overview of our further development of the paper [7] by Robert Downes, Michael Levitin and Dmitri Vassiliev and it will also give an insight how spectrum of massless Dirac operator on a 3-manifold interplays with geometric contents of the manifold. In contrast to the Riemann flat manifold studied in [7], 3-torus, we study the massless Dirac operator on a 3-sphere equipped with standard Riemannian metric. For the standard metric, the spectrum is known as the following.

\[ \pm \left( k + \frac{1}{2} \right), \quad k = 1, 2, \ldots, \]  

with multiplicity

\[ k(k + 1). \]  

We would give an asymptotic analysis of eigenvalues when the metric is perturbed, in an arbitrary fashion, from the standard one. To avoid the metric perturbation affecting our Hilbert space, we used the concept of massless Dirac operator on half-density that was introduced in the paper [6]. Note that in paper [6], the subprincipal symbol of massless Dirac operator has been studied in details and it is related to the Hodge dual of the axial torsion of teleparallel connection.

Physics behind our mathematical modeling

Assume that a spin-\( \frac{1}{4} \) massless particle is initially living in 3-sphere and its energy spectrum is given by the spectrum of our massless Dirac operator with standard metric. Now suppose there is a small fluctuation (perturbation) on

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the space i.e. a perturbation on the metric. We want to understand how the spectrum varies under the impact of such a fluctuation. In particular, we are interested in the lowest, in terms of modulus, positive and negative eigenvalues, which correspond to the lowest energy level of the particle and the anti-particle. The reason for this is that the asymptotic behavior of higher eigenvalues is studied in paper [6], which is also known as Weyl law for massless Dirac operators. This is also a special case of first order elliptic system studied in paper [5]. In short, our mathematical model is designed to answer the following questions.

(1) From the perturbed spectrum, can we get any information about the perturbed space where the particle or antiparticle lives?

(2) Does the energy levels of both particle and antiparticle shift symmetrically under such a small perturbation?

**Main results**

Consider a metric \( g_{\alpha\beta}(y; \epsilon) \) whose components are smooth functions of local coordinates \( y^\alpha, \alpha = 1, 2, 3 \), and small real parameter \( \epsilon \) and which turns into the standard metric \( (g_0)_{\alpha\beta}(y) \) for \( \epsilon = 0 \). Let \( \lambda_+(\epsilon) \) and \( \lambda_-(\epsilon) \) be the lowest, in terms of modulus, positive and negative eigenvalues of the Dirac operator \( W(\epsilon) \). Then we have the asymptotic expansions

\[
\lambda_\pm(\epsilon) = \pm \frac{3}{2} + \lambda_\pm^{(1)}(1) + \lambda_\pm^{(2)}(2) \epsilon^2 + O(\epsilon^3) \quad \text{as} \quad \epsilon \to 0. \tag{0.3}
\]

Note that \( \lambda_\pm(\epsilon) \) are double eigenvalues which cannot split because eigenvalues of the Dirac operator have even multiplicity, see Appendix A of [6] for details. Note also that the arguments presented in [7] apply to any double eigenvalue of the Dirac operator on any compact orientable Riemannian 3-manifold, so we know a priori that \( \lambda_\pm(\epsilon) \) admit the asymptotic expansions (0.3). Let

\[
V(\epsilon) := \int_{S^3} \sqrt{\det\{g_{\mu\nu}(y; \epsilon)\}} \ dy. \tag{0.4}
\]

Then

\[
V(\epsilon) = V^{(0)} + V^{(1)}(1) + O(\epsilon^2) \quad \text{as} \quad \epsilon \to 0, \tag{0.5}
\]

where

\[
V^{(0)} = \int_{S^3} \rho_0(y) \ dy = 2\pi^2 \tag{0.6}
\]

is the volume of the unperturbed sphere,

\[
\rho_0 := \sqrt{\det\{(g_0)_{\mu\nu}\}} \tag{0.7}
\]

is the standard Riemannian density on the sphere,

\[
V^{(1)} = \frac{1}{2} \int_{S^3} h_{\alpha\beta}(y) (g_0)^{\alpha\beta}(y) \rho_0(y) \ dy \tag{0.8}
\]
and
\[
h_{\alpha \beta} := \frac{\partial g_{\alpha \beta}}{\partial \epsilon} \bigg|_{\epsilon=0}.
\] (0.9)

**Theorem 0.1.** We have
\[
\lambda_{\pm}^{(1)} = \mp \frac{1}{4\pi^2} V^{(1)}.
\] (0.10)

This theorem partly answers our first question. Note that the dependence of the two lowest eigenvalues, \(\lambda_{\pm}(\epsilon)\), on the small parameter \(\epsilon\) is, in the first approximation, very simple: it is determined by the change of volume only. As expected, an increase of the volume of the resonator (volume of our Riemannian manifold) leads to a decrease of the two lowest natural frequencies (absolute values of the two lowest eigenvalues). Furthermore, if we denote by \(\lambda_{\pm}^{(0)} = \pm \frac{3}{2}\) the unperturbed values of the two lowest eigenvalues, then formula (0.10) can be rewritten as
\[
\frac{\lambda_{\pm}^{(1)}}{\lambda_{\pm}^{(0)}} = -\frac{1}{3} \frac{V^{(1)}}{V^{(0)}}.
\] (0.11)

Put \(\ell(\epsilon) := (V(\epsilon))^{1/3} = \ell(0) \left(1 + \frac{1}{3} \frac{V^{(1)}}{V^{(0)}} \epsilon + O(\epsilon^2)\right)\), where \(\ell(0) = \ell(0) = (2\pi^2)^{1/3}\). The quantity \(\ell(\epsilon)\) can be interpreted as the characteristic length of our Riemannian manifold. It is easy to see that formula (0.11) is equivalent to the statement
\[
\lambda_{\pm}(\epsilon) = \frac{\lambda_{\pm}^{(0)} \ell(\epsilon)}{\ell(\epsilon)} + O(\epsilon^2),
\] (0.12)
which shows that in the first approximation the two lowest eigenvalues are inversely proportional to the characteristic length.

The second question is an important topic in the spectral theory of first order elliptic systems, it is known as the topic of spectral asymmetry \[1, 2, 3, 4, 7\], i.e. asymmetry of the spectrum about zero. Formulae (0.3) and (0.10) imply
\[
\lambda_+(\epsilon) + \lambda_-(\epsilon) = \left(\lambda_+^{(2)} + \lambda_-^{(2)}\right) \epsilon^2 + O(\epsilon^3) \quad \text{as} \quad \epsilon \to 0,
\] (0.13)
which means that there is no spectral asymmetry in the first approximation in \(\epsilon\) but there may be spectral asymmetry in terms quadratic in \(\epsilon\).

We evaluate the asymptotic coefficients \(\lambda^{(2)}_{\pm}\) under the simplifying assumption that the Riemannian density does not depend on \(\epsilon\):
\[
\sqrt{\det\{g_{\mu \nu}(y; \epsilon)\}} = \rho_0(y).
\] (0.14)

In mechanics such a deformation is called shear \[8\]. Obviously, formula (0.14) implies \(\lambda_{\pm}^{(1)} = 0\), so formula (0.3) now reads
\[
\lambda_{\pm}(\epsilon) = \pm \frac{3}{2} + \lambda_{\pm}^{(2)} \epsilon^2 + O(\epsilon^3) \quad \text{as} \quad \epsilon \to 0.
\] (0.15)
Theorem 0.2. Under the assumption (0.14) we have

\[ \lambda^{(2)}_{\pm} = \pm \frac{1}{8\pi^2} \int_{S^3} Q_{\pm}(y) \rho_0(y) \, dy, \tag{0.16} \]

where

\[ Q_{\pm} = (h_{\pm})_{jk}(h_{\pm})_{jk} \]
\[ + \frac{1}{4} \varepsilon_{qks}(h_{\pm})_{jq} [(L_{\pm})_s(h_{\pm})_{jk}] \]
\[ - \frac{1}{8} (h_{\pm})_{ks} [(-\Delta)^{-1}(L_{\pm})_s(L_{\pm})_j(h_{\pm})_{jk}] \]
\[ + \frac{1}{4} \varepsilon_{qks}(h_{\pm})_{rq} [(-\Delta)^{-1}(L_{\pm})_r(L_{\pm})_s(L_{\pm})_j(h_{\pm})_{jk}] \]. \tag{0.17}

Here \( \varepsilon_{qrs} \) is the totally antisymmetric quantity, \( \varepsilon_{123} := +1 \).

Theorem 0.2 warrants the several remarks and these remarks will be present in the talk. In particular one of the remarks gives a solution to the second question, which leads us to expect spectral asymmetry in terms quadratic.

References


