

Axial Torsion Waves in Metric–affine Gravity

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We construct new explicit vacuum solutions of quadratic metric-affine gravity. The approach of metric-affine gravity in using an independent affine connection produces a theory with 10+64 unknowns, which implies admitting torsion and possible nonmetricity. Our spacetimes are generalisations of classical pp-waves, four-dimensional Lorentzian spacetimes which admit a nonvanishing parallel spinor field. We generalize this definition to metric compatible spacetimes with pp-metric and purely axial torsion. It has been suggested that one can interpret that the axial component of torsion as the Hodge dual of the electromagnetic vector potential. We compare these solutions with our previous results and other solutions of classical models describing the interaction of gravitational and neutrino fields.

Keywords: Metric–affine gravity, Torsion waves, PP-waves, Yang–Mills theory, Einstein–Weyl theory

1. Introduction

Spacetime is considered to be a connected real 4-manifold M equipped with a Lorentzian metric g and an affine connection Γ . The unknowns are the 10 independent components of the metric tensor $g_{\mu\nu}$ and the 64 connection coefficients $\Gamma^\lambda_{\mu\nu}$. This alternative theory of gravity is called *metric–affine gravity*. In *quadratic metric–affine gravity* (QMAG), we define our action as $S := \int q(R)$ where q is an $O(1,3)$ -invariant quadratic form on curvature R . Independently varying the action with respect to the metric g and the connection Γ produces the system of Euler–Lagrange equations which we will write symbolically as

$$\partial S / \partial g = 0 \tag{1}$$

$$\partial S / \partial \Gamma = 0. \tag{2}$$

Our objective is to study of the system (1), (2). Our motivation comes from Yang–Mills theory. The Yang–Mills action for the affine connection is a special case of where $q(R) = R^\kappa_{\lambda\mu\nu} R^\lambda_{\kappa}{}^{\mu\nu}$. The results in this paper rely on the work of C.N. Yang¹ and E.W. Mielke² who respectively showed that Einstein spaces satisfy equations (2) and (1) in the Yang–Mills case. Our notation follows Refs. 3, 4, 5, 6, 7, 8, 9.

2. Generalising pp-waves to spacetimes with axial torsion

In constructing our solutions of the system (1), (2) we use classical pp-waves, see Refs. 4, 8 and extensive references therein for much more information on pp-waves and known pp-wave type solutions of metric–affine gravity. We define a *pp-wave* as a

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Riemannian spacetime which admits a nonvanishing parallel spinor field. Vassiliev³ showed that pp-waves of parallel Ricci curvature are solutions of the system of equations (1), (2). We denote the nonvanishing parallel spinor field by $\chi = \chi^a$ and we assume this spinor field to be *fixed*. Put $l^\alpha := \sigma^\alpha_{ab} \chi^a \bar{\chi}^b$ where σ^α are Pauli matrices. Then l is a nonvanishing parallel real null vector field. We define the real scalar function, which we call the *phase*, as $\varphi : M \rightarrow \mathbb{R}$, $\varphi(x) := \int l \cdot dx$.

Put $F_{\alpha\beta} := \sigma_{\alpha\beta ab} \chi^a \chi^b$ where $\sigma_{\alpha\beta ac} := \frac{1}{2}(\sigma_{\alpha ab} \epsilon^{bd} \sigma_{\beta cd} - \sigma_{\beta ab} \epsilon^{bd} \sigma_{\alpha cd})$. F can be expressed as $F = l \wedge m$ where m is a complex vector field satisfying $m_\alpha m^\alpha = l_\alpha m^\alpha = 0$, $m_\alpha \bar{m}^\alpha = -2$. It is known that pp waves can also be defined as a Riemannian spacetime whose metric can be written locally in the form

$$ds^2 = 2 dx^0 dx^3 - (dx^1)^2 - (dx^2)^2 + f(x^1, x^2, x^3) (dx^3)^2 \quad (3)$$

in some local coordinates (x^0, x^1, x^2, x^3) . The curvature tensor R is linear in f , i.e. $R_{\alpha\beta\gamma\delta} = -\frac{1}{2}(l \wedge \partial)_{\alpha\beta} (l \wedge \partial)_{\gamma\delta} f$ where $(l \wedge \partial)_{\alpha\beta} := l_\alpha \partial_\beta - \partial_\alpha l_\beta$. We will restrict our choice to those coordinates in which

$$\chi^a = (1, 0), \quad l^\mu = (1, 0, 0, 0), \quad m^\mu = (0, 1, \mp i, 0). \quad (4)$$

With such a choice the phase function becomes $\varphi(x) = x^3 + \text{const.}$ the curvature of a pp-wave can be rewritten in invariant form, i.e. $R = -\frac{1}{2}(l \wedge \nabla) \otimes (l \wedge \nabla) f$, where $l \wedge \nabla := l \otimes \nabla - \nabla \otimes l$. The curvature of a pp-wave has the two irreducible pieces, namely (symmetric) trace-free Ricci and Weyl, see Ref. 7.

Definition 2.1. A generalised pp-wave with purely axial torsion is a metric compatible spacetime with pp-metric and torsion

$$T := *A \quad (5)$$

where A is a real vector field defined by $A = k(\phi)l$, where $k : \mathbb{R} \mapsto \mathbb{R}$ is an arbitrary real function of the phase.

Note that the real vector field A is a plane wave solution of the polarized Maxwell equation $*dA = \pm idA$. It has been suggested, see Refs. 9, 10, that one can interpret the axial component of torsion as the Hodge dual of the electromagnetic vector potential. We can express torsion as $T = \mp \frac{i}{2} k(x^3) l \wedge m \wedge \bar{m}$, using our special local coordinates (3), (4).

Torsion (5) is clearly purely axial, as $T_{\kappa\mu\nu} = k(\varphi) l^\lambda \epsilon_{\lambda\kappa\mu\nu}$, i.e. the totally antisymmetric part of torsion.

Remark 2.1. Torsion (5) corresponds to the axial torsion first obtained by Singh^{11,12}. Put $m = -\frac{1}{2}k(x^3)$ in formula (16) of Ref. 11 or put $n = 0, m = -\frac{1}{2}k(x^3)$ in formula (20) of Ref. 12.

Note that the connection of a generalised pp-wave with purely axial torsion is *metric compatible*. The curvature of a generalised pp-wave has a very nice explicit formula

$$R = -\frac{1}{2}(l \wedge \{\nabla\}) \otimes (l \wedge \{\nabla\})f + \frac{1}{4}k(\varphi)^2 \text{Re}((l \wedge m) \otimes (l \wedge \bar{m})) \mp \frac{1}{2}k'(\varphi) \text{Im}((l \wedge m) \otimes (l \wedge \bar{m})) \quad (6)$$

It is a highly nontrivial fact that the torsion generated curvature and the Riemannian curvature simply add up to produce formula (6). Ricci curvature now reads $Ric = \frac{1}{2}(f_{11} + f_{22} - k^2) l \otimes l$. Scalar curvature is clearly zero. Ricci curvature is parallel, i.e. $\nabla Ric = 0$, if $f_{11} + f_{22} = k^2 + C$, in which case $Ric = \Lambda l \otimes l$, for some constant Λ .

3. New solutions of quadratic metric-affine gravity

Theorem 3.1. *Generalised pp-waves with purely axial torsion of parallel $\{Ric\}$ are solutions of (1), (2) in the Yang–Mills case.*

Note that by $\{Ric\}$ we denote the Ricci curvature generated by the Levi-Civita connection only. The condition $\{\nabla\}\{Ric\} = 0$ implies that $f_{11} + f_{22} = C$. Note that the result also holds if Ric is assumed to be parallel.

The torsion generated curvature has two pieces of curvature, namely $R^{(1)}$ and $R^{(5)}$ as denoted in Ref. 6, or ${}^{(2)}W_{\alpha\beta}$ and ${}^{(4)}W_{\alpha\beta}$ as denoted in Ref. 13, or ${}^{(2)}R^{\alpha\beta}$ (Paircom (9)) and ${}^{(4)}R^{\alpha\beta}$ (Ricsymf (9)), as denoted in Ref. 14.

Remark 3.1. It can be shown that for an arbitrary purely axial torsion, the Weyl curvature of the resulting torsion generated curvature is zero.

Using notation from Refs. 3, 4, 5, 8, the irreducible subspaces from which our torsion generated curvature hails are $\mathbf{R}^{(9,l)}$ and $\mathbf{R}_*^{(9,l)}$ which can be easily shown by using the explicit formula for the right Hodge dual of torsion generated curvature

$$(R_T)^* = \pm \frac{1}{4}k(x^3)^2 \text{Im}((l \wedge m) \otimes (l \wedge \bar{m})) + \frac{1}{2}k'(x^3) \text{Re}((l \wedge m) \otimes (l \wedge \bar{m}))$$

Now we want to see whether our spacetimes are solutions of the problem (1), (2) for the most general quadratic form, see Refs. 3, 4. Our main result is the following

Theorem 3.2. *Generalised pp-waves with purely axial torsion of parallel $\{Ric\}$ are solutions of the system of equations (1), (2).*

4. Discussion and physical interpretation

The solution obtained in Section 2 is a completely new solution of quadratic metric-affine gravity and it differs substantially from our previous results obtained in Refs. 4, 8, where torsion is purely tensor, as well as those from Ref. 11, where the metric is different and the author only considers the Yang–Mills case. Blagojević and Hehl¹⁴ provide a review of other exact solutions in this theory.

Similarly to the approach of Refs. 4, 8, we wish to compare these solutions to the classical models describing the interaction of gravitational and massless neutrino fields, namely Einstein–Weyl theory. Our torsion and the corresponding torsion generated curvature are waves traveling at speed of light because k is a function of the phase φ which has the role of a null coordinate, $g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi=0$. Note that our model differs from the classical in that we have conformal invariance, while the classical models contain the gravitational constant. We choose to deal with the complexified curvature $\mathfrak{R}:=r(l\wedge m)\otimes(l\wedge\bar{m})$, where $r:=\frac{1}{4}k^2-\frac{i}{2}k'$. This complexification is in line with traditions of quantum mechanics. The complexified curvature is polarized, i.e. ${}^*\mathfrak{R}=\mathfrak{R}^*=\pm i\mathfrak{R}$, and equivalent to a (symmetric) rank 4 spinor ω , and we have that $\mathfrak{R}_{\alpha\beta\gamma\delta}=\sigma_{\alpha\beta ab}\omega^{abcd}\bar{\sigma}_{\gamma\delta cd}$, where $\sigma_{\alpha\beta}$ are the second order Pauli matrices from Section 2. Using the properties of pp-waves, we get that $\omega=\xi\otimes\xi\otimes\xi\otimes\xi$, where $\xi:=r^{1/4}\chi$ and χ is the spinor from Section 2.

We claim that the spinor field ξ satisfies the massless Dirac equation $\sigma^\mu{}_{ab}\nabla_\mu\xi^a=0$, see Ref. 4. Furthermore, as was similarly done in Ref. 4, we can construct pp-wave type solutions of Einstein–Weyl theory very similar to purely axial torsion pp-wave solutions of QMAG from Section 3 to conclude that generalised pp-waves represent a metric-affine model for the massless neutrino.

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